

# Strong-coupling expansion for the spin-1 Bose-Hubbard model

Takashi Kimura\*

*Department of Mathematics and Physics,  
Kanagawa University, 2946 Tsuchiya,  
Hiratsuka, Kanagawa 259-1293, Japan*

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## Abstract

We perform a strong-coupling expansion up to the third order of the hopping parameter  $t$  for the spin-1 Bose-Hubbard model with antiferromagnetic interaction. We examine the validity of the expansion by comparing it with the mean-field theory. Moreover, we apply a few extrapolation methods up to the infinite order of  $t$  to improve our results.

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\* tkimura@kanagawa-u.ac.jp

## I. INTRODUCTION

Ultracold bosons have been studied extensively since the realization of the Bose-Einstein condensation. Here temperature can be approximately zero, which is very difficult to realize in conventional experimental systems. In addition to the conventional spinless bosons, spinor bosons have been examined as a new bosonic system with the multiple internal degrees of freedom [1, 2]. The development of the optical lattice systems has further promoted the study of ultracold bosons. In particular, the superfluid (SF)-Mott insulator (MI) transition has been realized in an optical lattice system [3–7].

Theoretically, the optical-lattice system having low boson fillings can generally be described by the Bose-Hubbard (BH) model [8, 9]. In addition, both the MI phases and SF-MI transitions of spin-1 bosons have been intensively studied analytically [10–28]. The ferromagnetically interacting system is essentially similar to spinless bosons, whereas the antiferromagnetically interacting system has rich physical properties. Several spin phases such as the singlet, nematic, and dimerized phases in the insulating phase have been examined analytically [10–14] and numerically [22–25]. As for the SF-MI transition, Tsuchiya, Kurihara and the present author [16] have used perturbative mean-field approximation (PMFA) [29], which expands the free energy in the SF-order parameter, to quantitatively show that the MI phase for even boson fillings is considerably stable against the SF phase than that for odd boson fillings. This conjecture has been confirmed by the density-matrix-renormalization group (DMRG) [22] and quantum-Monte-Carlo simulation (QMC) [24, 25] in one dimension (1D). On the other hand, mean-field (MF) studies beyond the perturbation theory [17–19] have shown a possible first-order SF-MI transition of the BH model by MF theory for a weak antiferromagnetic interaction such as  $U_2/U_0 \simeq 0.04$  corresponding to  $^{23}\text{Na}$ . The first-order transition in 1D has also been shown by a QMC study [24]. On the other hand, if the antiferromagnetic interaction is adequately strong, the first-order transition may be neglected because it occurs when kinetic energy is considerably greater than antiferromagnetic interaction energy near the SF-MI phase boundary [30]. For a second-order transition, the strong-coupling expansion method [31] of the kinetic energy, which is based on the Rayleigh-Schrödinger perturbation theory [32], is an excellent method for obtaining the phase boundary. The strong-coupling expansion has been applied in the analysis of the spinless BH [31, 33–37], extended BH [38], hardcore BH [39], and two-species BH models [40], and the

results are in excellent agreement with QMC results [36, 39]. So far, however, only MF calculations have been performed to analytically study the SF-MI transition of the spin-1 BH model.

In this study, we perform a strong-coupling expansion of the spin-1 BH model up to the third order of the hopping parameter  $t$ . This paper is organized as follows: Section II introduces the spin-1 BH model and strong-coupling expansion. Section III gives the results: the phase diagrams for two dimensions (2Ds) or three dimensions (3Ds), the critical values of  $t$  at the peak of the Mott lobes and their dependence on antiferromagnetic interaction, which reflect the validity of the expansion, results obtained by several extrapolation techniques up to the infinite order of  $t$ , and the 1D phase diagram. A brief summary of our results and discussions are given in Sec. V.

## II. SPIN-1 BOSE-HUBBARD MODEL

The spin-1 BH model is given by  $H = H_0 + H_1$ ,

$$\begin{aligned}
H_0 &= -t \sum_{\langle i,j \rangle, \alpha} (a_{i\alpha}^\dagger a_{j\alpha} + a_{j\alpha}^\dagger a_{i\alpha}), \\
H_1 &= -\mu \sum_{i, \alpha} a_{i\alpha}^\dagger a_{i\alpha} + \frac{1}{2} U_0 \sum_{i, \alpha, \beta} a_{i\alpha}^\dagger a_{i\beta}^\dagger a_{i\beta} a_{i\alpha} \\
&\quad + \frac{1}{2} U_2 \sum_{i, \alpha, \beta, \gamma, \delta} a_{i\alpha}^\dagger a_{i\gamma}^\dagger \mathbf{F}_{\alpha\beta} \cdot \mathbf{F}_{\gamma\delta} a_{i\delta} a_{i\beta}. \\
&= \sum_i \left[ -\mu \hat{n}_i + \frac{1}{2} U_0 \hat{n}_i (\hat{n}_i - 1) + \frac{1}{2} U_2 (\hat{\mathbf{S}}_i^2 - 2\hat{n}_i) \right].
\end{aligned} \tag{1}$$

Here,  $\mu$  and  $t(>0)$  are the chemical potential and the hopping matrix element, respectively.  $U_0$  ( $U_2$ ) represents the spin-independent (dependent) interaction between bosons. We assume  $U_0$  and  $U_2$  are positive, which corresponds to repulsive and antiferromagnetic interaction. The operator  $a_{i\alpha}$  ( $a_{i\alpha}^\dagger$ ) annihilates (creates) a boson at site  $i$  with spin-magnetic quantum number  $\alpha = 1, 0, -1$ . The number operator at site  $i$  is given by  $n_i \equiv \sum_\alpha n_{i\alpha}$  ( $n_{i\alpha} \equiv a_{i\alpha}^\dagger a_{i\alpha}$ ).  $\hat{\mathbf{S}}_i \equiv \sum_{\alpha, \beta} a_{i\alpha}^\dagger \mathbf{F}_{\alpha\beta} a_{i\beta}$  is a spin operator at site  $i$  and  $\mathbf{F}_{\alpha\beta}$  represents the spin-1 matrices. In this study, we assume a tight binding model with only nearest-neighbor hopping. The summation for all the sets of adjacent sites is expressed by  $\langle i, j \rangle$ . For simplicity, we assume hypercubic lattices.

Under the limit of  $t \rightarrow 0$ , the MI phase exists for arbitrary  $\mu$ ; the MI phase also has even  $n_0$  bosons per site for  $U_0(n_0 - 1) - 2U_2 < \mu < U_0 n_0$  or odd  $n_0$  boson(s) for  $U_0 n_0 < \mu < U_0 n_0 + 2U_2$ . To ensure that the phase diagram has MI phases with odd boson(s) per site, we assume  $U_0 > 2U_2 > 0$ . The SF-MI phase boundary can be determined by calculating the energy of the MI phase and that of their defect state which has exactly one extra particle or hole. Specifically, if  $E_{\text{MI}}(n_0, \mu, t) > (<) E^{\text{part}}(n_0, \mu, t)$  or  $E_{\text{MI}}(n_0, \mu, t) > (<) E^{\text{hole}}(n_0, \mu, t)$ , we determine that the phase is SF (MI), where  $E_{\text{MI}}(n_0, \mu, t)$  is the energy of the MI state and  $E^{\text{part}}(n_0, \mu, t)$  [ $E^{\text{hole}}(n_0, \mu, t)$ ] is the energy of the defect state with one extra particle (hole). The SF-MI phase boundary is determined by

$$E_{\text{MI}}(n_0, \mu, t) = E^{\text{part}}(n_0, \mu, t) \quad (2)$$

or

$$E_{\text{MI}}(n_0, \mu, t) = E^{\text{hole}}(n_0, \mu, t). \quad (3)$$

### III. STRONG-COUPLING EXPANSION

By following Ref. [31], we employ the Rayleigh-Schrödinger perturbation theory for the third order of the hopping parameter  $t$ . We start from the unperturbed MI states, define the defect state by doping an extra particle or hole to the MI states, and compare the energy of the MI state with that of the defect state.

#### A. MI states at the zeroth order of the hopping parameter

The unperturbed MI state with even bosons per site is

$$\Psi_{\text{even}} = \prod_k |n_0, 0, 0\rangle_k, \quad (4)$$

where  $|n_0, 0, 0\rangle_k$  implies the boson number  $N = n_0$ , the spin magnitude is  $S = 0$ , and the spin-magnetic quantum number is  $S_z = 0$  at site  $k$ . For simplicity, we neglect the nematic MI state including  $S = 2$  states. However, from analytical calculations [12, 13] and a recent 2D variational Monte-Carlo study [26], the nematic MI phase for even boson filling may exist for a weak  $U_2$ .

For the unperturbed MI state with odd bosons per site, we assume a nematic MI state

$$\Psi_{\text{odd}} = \prod_k |n_0, 1, 0\rangle_k. \quad (5)$$

Although  $\Psi_{\text{ferro}} = \prod_k |n_0, 1, \pm 1\rangle_k$  is degenerate with  $\Psi_{\text{odd}}$  at  $t = 0$ , we can easily find that the degeneracy is lifted at finite  $t$ , and  $\Psi_{\text{odd}}$  has lower energy at least in the third-order perturbation of  $t$  as expected (Appendix A). This is natural because we assume antiferromagnetic interaction. At  $t = 0$ , a dimerized state

$$\begin{aligned} \Psi_{\text{dimer}} = \prod_{\langle i,j \rangle} \frac{1}{\sqrt{3}} & \left( |n_0, 1, 1\rangle_i \otimes |n_0, 1, -1\rangle_j \right. \\ & \left. - |n_0, 1, 0\rangle_i \otimes |n_0, 1, 0\rangle_j + |n_0, 1, -1\rangle_i \otimes |n_0, 1, 1\rangle_j \right), \end{aligned} \quad (6)$$

is also degenerate with  $\Psi_{\text{odd}}$ , where we assume that  $n_0 - 1$  bosons form a spin singlet in each site and extra bosons form dimers between adjacent sites. The dimerized state is considered to be the ground state in 1D [11, 12, 14, 22–25]. Therefore, the validity of our results based on  $\Psi_{\text{odd}}$  for odd boson fillings is basically limited to 2D or 3D systems, although the dimerized phase cannot be denied even there. We can easily find that the energy of  $\Psi_{\text{dimer}}$  is greater than that of  $\Psi_{\text{odd}}$  in the third-order perturbation of  $t$  in 2D or 3D (Appendix A). We note that  $\Psi_{\text{even}}$  ( $\Psi_{\text{odd}}$ ) is also adopted as the ground-state in PMFA [16] which is compared with our results in the following section.

## B. Defect states

We define the defect states by doping an extra particle or hole to  $\Psi_{\text{even}}$  and  $\Psi_{\text{odd}}$  as follows:

$$\Psi_{\text{even}}^{\text{part}} = \frac{1}{\sqrt{N}} \sum_i \left[ f_i |n_0 + 1, 1, 0\rangle_i \otimes \prod_{k \neq i} |n_0, 0, 0\rangle_k \right], \quad (7)$$

$$\Psi_{\text{even}}^{\text{hole}} = \frac{1}{\sqrt{N}} \sum_i \left[ f_i |n_0 - 1, 1, 0\rangle_i \otimes \prod_{k \neq i} |n_0, 0, 0\rangle_k \right], \quad (8)$$

$$\Psi_{\text{odd}}^{\text{part}} = \frac{1}{\sqrt{N}} \sum_i \left[ f_i |n_0 + 1, 0, 0\rangle_i \otimes \prod_{k \neq i} |n_0, 1, 0\rangle_k \right], \quad (9)$$

$$\Psi_{\text{odd}}^{\text{hole}} = \frac{1}{\sqrt{N}} \sum_i \left[ f_i |n_0 - 1, 0, 0\rangle_i \otimes \prod_{k \neq i} |n_0, 1, 0\rangle_k \right]. \quad (10)$$

Here  $N$  is the number of lattice sites, and  $f_i$  is the eigenvector of the hopping matrix  $t_{ij}$  with the highest eigenvalue [31]. In this study, because we assume hypercubic lattices

with only nearest-neighbor hopping,  $f_i = 1$  and the eigenvalue  $\lambda = zt$ , where  $z = 2d$  is the number of nearest-neighbor sites in the  $d$ -dimensional hypercubic lattice. Although,  $\Psi_{\text{odd}}^{\text{part(hole)}}$  has no other degenerate candidates,  $\Psi_{\text{even}}^{\text{part(hole)}}$  is degenerate with  $\Theta_{\pm}^{\text{part(hole)}} = \frac{1}{\sqrt{N}} \sum_i [|n_0 + (-)1, 1, \pm 1\rangle_i \otimes \prod_{k \neq i} |n_0, 1, 0\rangle_k]$ . We find that  $\Theta_{\pm}^{\text{part(hole)}}$  has the exact same energy with  $\Psi_{\text{even}}^{\text{part(hole)}}$  up to the third order of  $t$ , and we can choose  $\Psi_{\text{even}}^{\text{part(hole)}}$  as the defect state. This degeneracy will be lifted when we dope particles or holes with finite density because the SF phase will have no magnetization.

### C. Ground-state energies and phase diagrams

By using  $\Psi_{\text{even}}$  and  $\Psi_{\text{odd}}$ , the energies of the MI state with even or odd boson(s) per site are obtained as

$$\begin{aligned} \frac{E_{\text{MI,even}}(n_0)}{N} &= \frac{U_0}{2} n_0(n_0 - 1) - U_2 n_0 - n_0 \mu - \frac{zt^2}{3} \frac{n_0(n_0 + 3)}{U_0 + 2U_2} \\ \frac{E_{\text{MI,odd}}(n_0)}{N} &= \frac{U_0}{2} n_0(n_0 - 1) - U_2(n_0 - 1) - n_0 \mu \\ &\quad - zt^2 \left[ \frac{34}{225} \frac{(n_0 + 4)(n_0 - 1)}{U_0 + 4U_2} + \frac{4}{45} \frac{2n_0^2 + 6n_0 + 7}{U_0 + U_2} + \frac{1}{9} \frac{(n_0 + 1)(n_0 + 2)}{U_0 - 2U_2} \right] \end{aligned} \quad (11)$$

up to the third order of  $t$ . On the other hand, the energies of the defect states are obtained as

$$\begin{aligned} &E_{\text{def,even}}^{\text{part}}(n_0) - E_{\text{MI,even}}(n_0) \\ &= U_0 n - \mu - zt \frac{n_0 + 3}{3} \\ &\quad - \frac{z(z-7)t^2}{9} \frac{n_0(n_0 + 3)}{U_0 + 2U_2} - \frac{zt^2 n_0}{9} \left[ 2 \left( \frac{n_0 + 5}{2U_0 + 3U_2} + \frac{n_0 + 3}{3U_2} \right) + \frac{n_0 + 2}{2U_0} \right] \\ &\quad - \frac{zt^3}{27} n_0(n_0 + 3) \left\{ (z-1) \left[ \frac{(2n_0 + 3)z - 3(3n_0 + 8)}{(U_0 + 2U_2)^2} \right. \right. \\ &\quad \left. \left. + \frac{2}{U_0 + 2U_2} \left( 2 \frac{n_0 + 5}{2U_0 + 3U_2} + \frac{n_0 + 2}{2U_0} \right) + \frac{4(n_0 + 3)}{3U_2(U_0 + 2U_2)} \right] \right. \\ &\quad \left. - z \left[ 2 \left( \frac{n_0 + 5}{(2U_0 + 3U_2)^2} + \frac{n_0 + 3}{(3U_2)^2} \right) + \frac{n_0 + 2}{(2U_0)^2} \right] + \frac{4}{3U_2} \left( \frac{1}{5} \frac{n_0 + 5}{2U_0 + 3U_2} + \frac{n_0 + 2}{2U_0} \right) \right\}, \end{aligned} \quad (13)$$

$$\begin{aligned}
& E_{\text{def,even}}^{\text{hole}}(n_0) - E_{\text{MI,even}}(n_0) \\
&= -U_0(n_0 - 1) + 2U_2 + \mu - zt \frac{n_0}{3} \\
&\quad - \frac{z(z-7)t^2}{9} \frac{n_0(n_0+3)}{U_0+2U_2} - \frac{zt^2(n_0+3)}{9} \left[ 2 \left( \frac{n_0-2}{2U_0+3U_2} + \frac{n_0}{3U_2} \right) + \frac{n_0+1}{2U_0} \right] \\
&\quad - \frac{zt^3}{27} n_0(n_0+3) \left\{ (z-1) \left[ \frac{(2n_0+3)z - 3(3n_0+1)}{(U_0+2U_2)^2} \right. \right. \\
&\quad \left. \left. + \frac{2}{U_0+2U_2} \left( 2 \frac{n_0-2}{2U_0+3U_2} + \frac{n_0+1}{2U_0} \right) + \frac{4n_0}{3U_2(U_0+2U_2)} \right] \right. \\
&\quad \left. - z \left[ 2 \left( \frac{n_0-2}{(2U_0+3U_2)^2} + \frac{n_0}{(3U_2)^2} \right) + \frac{n_0+1}{(2U_0)^2} \right] + \frac{4}{3U_2} \left( \frac{1}{5} \frac{n_0-2}{2U_0+3U_2} + \frac{n_0+1}{2U_0} \right) \right\}, \quad (14)
\end{aligned}$$

$$\begin{aligned}
& E_{\text{def,odd}}^{\text{part}}(n_0) - E_{\text{MI,odd}}(n_0) \\
&= U_0 n_0 - 2U_2 - zt \frac{n_0+1}{3} - \mu \\
&\quad - \frac{z(z-3)t^2}{9} \left[ \frac{(n_0+1)(n_0+2)}{U_0-2U_2} + \frac{4}{5} \frac{n_0^2-1}{U_0+U_2} \right] \\
&\quad + \frac{4}{45} zt^2 \left[ 2 \frac{(n_0+2)(n_0+4)}{U_0+U_2} + \frac{17}{5} \frac{(n_0-1)(n_0+4)}{U_0+4U_2} \right] \\
&\quad - \frac{zt^2}{9} \left[ 2 \frac{(n_0-1)(n_0+4)}{2U_0+3U_2} + \frac{(n_0+2)(n_0+4)}{2U_0} \right] - \frac{2}{45} z(2z+3)t^2 \frac{(n_0+1)(n_0+4)}{3U_2} \\
&\quad - \frac{z(z-1)t^3}{27} \frac{(n_0+1)(n_0+2)}{(U_0-2U_2)^2} [(2n_0+3)z - (5n_0+6)] \\
&\quad - \frac{4}{675} z(z-1)t^3 \frac{n_0+1}{(U_0+U_2)^2} [(n_0-1)(9n_0+1)z - 2(17n_0^2+26n_0+32)] \\
&\quad - \frac{z(z-1)^2 t^3}{27} \frac{n_0+1}{U_0+U_2} \left[ \frac{32}{25} \frac{(n_0-1)(n_0+4)}{U_0+4U_2} + \frac{8}{5} \frac{(n_0+2)(2n_0+3)}{U_0-2U_2} \right] \\
&\quad - \frac{2}{27} z(z-1)t^3 (n_0+1)(n_0+4) \left\{ \frac{n_0-1}{2U_0+3U_2} \left[ \frac{34}{25} \frac{1}{U_0+4U_2} + \frac{4}{5} \frac{1}{U_0+U_2} \right] \right. \\
&\quad \left. + \frac{n_0+2}{2U_0} \left[ \frac{1}{U_0-2U_2} + \frac{4}{5} \frac{1}{U_0+U_2} \right] + \frac{1}{3U_2} \left[ \frac{2}{25} (8z+9) \frac{n_0-1}{U_0+4U_2} + \frac{4}{5} z \frac{n_0+2}{U_0+U_2} \right] \right\} \\
&\quad - \frac{4}{135} z(2z+3)t^3 \frac{(n_0+1)(n_0+4)}{3U_2} \left[ \frac{1}{5} \frac{n_0-1}{2U_0+3U_2} + \frac{n_0+2}{2U_0} \right] \\
&\quad + \frac{zt^3}{675} \frac{(n_0+1)(n_0+4)}{(3U_2)^2} [4(n_0-11)z^2 + 18(3n_0+7)z - 15(n_0+4)] \\
&\quad + \frac{zt^3}{27} (n_0+1)(n_0+4) \left[ \frac{68}{25} (z-1) \frac{n_0-1}{(U_0+4U_2)^2} + 2z \frac{n_0-1}{(2U_0+3U_2)^2} + z \frac{n_0+2}{(2U_0)^2} \right], \quad (15)
\end{aligned}$$

$$\begin{aligned}
& E_{\text{def,odd}}^{\text{hole}}(n_0) - E_{\text{MI,odd}}(n_0) \\
&= -U_0(n_0 - 1) - zt \frac{n_0 + 2}{3} + \mu \\
&\quad - \frac{z(z-3)t^2}{9} \left[ \frac{(n_0+1)(n_0+2)}{U_0 - 2U_2} + \frac{4}{5} \frac{(n_0+2)(n_0+4)}{U_0 + U_2} \right] \\
&\quad + \frac{4}{45} zt^2 \left[ 2 \frac{n_0^2 - 1}{U_0 + U_2} + \frac{17}{5} \frac{(n_0-1)(n_0+4)}{U_0 + 4U_2} \right] \\
&\quad - \frac{zt^2}{9} \left[ 2 \frac{(n_0-1)(n_0+4)}{2U_0 + 3U_2} + \frac{n_0^2 - 1}{2U_0} \right] - \frac{2}{45} z(2z+3)t^2 \frac{(n_0-1)(n_0+2)}{3U_2} \\
&\quad - \frac{z(z-1)t^3}{27} \frac{(n_0+1)(n_0+2)}{(U_0 - 2U_2)^2} [(2n_0+3)z - (5n_0+9)] \\
&\quad - \frac{4}{675} z(z-1)t^3 \frac{n_0+2}{(U_0+U_2)^2} [(n_0+4)(9n_0+26)z - 2(17n_0^2+76n_0+107)] \\
&\quad - \frac{z(z-1)t^3}{27} \frac{n_0+2}{U_0+U_2} \left[ \frac{32}{25} \frac{(n_0-1)(n_0+4)}{U_0+4U_2} + \frac{8}{5} \frac{(n_0+1)(2n_0+3)}{U_0-2U_2} \right] \\
&\quad - \frac{2}{27} z(z-1)t^3 (n_0-1)(n_0+2) \left\{ \frac{n_0+4}{2U_0+3U_2} \left[ \frac{34}{25} \frac{1}{U_0+4U_2} + \frac{4}{5} \frac{1}{U_0+U_2} \right] \right. \\
&\quad \left. + \frac{n_0+1}{2U_0} \left[ \frac{1}{U_0-2U_2} + \frac{4}{5} \frac{1}{U_0+U_2} \right] + \frac{1}{3U_2} \left[ \frac{2}{25} (8z+9) \frac{n_0+4}{U_0+4U_2} + \frac{4}{5} z \frac{n_0+1}{U_0+U_2} \right] \right\} \\
&\quad - \frac{4}{135} z(2z+3)t^3 \frac{(n_0-1)(n_0+2)}{3U_2} \left[ \frac{1}{5} \frac{n_0+4}{2U_0+3U_2} + \frac{n_0+1}{2U_0} \right] \\
&\quad + \frac{zt^3}{675} \frac{(n_0-1)(n_0+2)}{(3U_2)^2} [4(n_0+14)z^2 + 18(3n_0+2)z - 15(n_0-1)] \\
&\quad + \frac{zt^3}{27} (n_0-1)(n_0+2) \left[ \frac{68}{25} (z-1) \frac{n_0+4}{(U_0+4U_2)^2} + 2z \frac{n_0+4}{(2U_0+3U_2)^2} + z \frac{n_0+1}{(2U_0)^2} \right] \quad (16)
\end{aligned}$$

up to the third order of  $t$ . By equating the right-hand side of Eqs. 13–16 to zero, we obtain the SF-MI phase boundary  $t$ – $\mu$  curve  $\mu_{\text{even}}^{\text{part}}(t)$ ,  $\mu_{\text{even}}^{\text{hole}}(t)$ ,  $\mu_{\text{odd}}^{\text{part}}(t)$ , and  $\mu_{\text{odd}}^{\text{hole}}(t)$ , which show the upper branch (corresponding to particle-doping) or lower branch (corresponding to hole-doping) of the phase boundary curve around the Mott phase with even boson fillings, and those for odd boson fillings, respectively.

Figures 1–4 show the phase diagram obtained by the calculation. The MI phase for even boson fillings is considerably stable against the SF phase than that for odd boson fillings as expected from MF and QMC studies. The area of the MI for even (odd) boson fillings is more increased (decreased) for  $U_2/U_0 = 0.3$  than that for  $U_2/U_0 = 0.15$ . The critical value of  $t$  on the phase boundary curve is greater in 2D than that in 3D, because the number of nearest-neighbor site  $z$  and kinetic energy are smaller at the same value of  $t$ . These curves



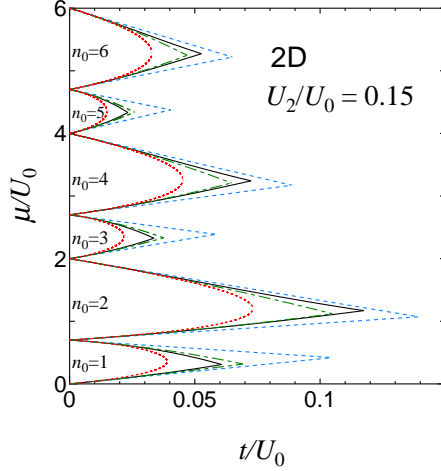


FIG. 1. (Color online) Phase diagram obtained by strong-coupling expansion (Eqs. 13–16) for  $U_2/U_0 = 0.15$  in 2D. The solid curves show the results up to the third order of  $t$ . Results up to the first order (second order) of  $t$  are also shown by the blue dashed (green dot-dashed) curve. The red dotted curve shows the results obtained by PMFA.

show the convergence of the strong-coupling expansion from the first to the third order of  $t$ , and we find convergence is excellent except for that in Fig. 1 for a weak  $U_2 = 0.15U_0$  in 2D.

Moreover, we plot the results of PMFA; the plot is exact in infinite dimensions as long as the SF-MI transition is of the second order. The area of the Mott lobes obtained by strong-coupling expansion are greater than those of PMFA. This difference may reflect the quantum fluctuations in the MI phases, which are incorporated (neglected) in strong-coupling expansion (PMFA).

#### D. Consistency in PMFA

PMFA involves MF decoupling theory using perturbation expansion of the SF order parameter. Up to a low-order expansion of the SF order parameter in Ref. [16], the first-order SF-MI transition cannot be predicted. However, if the SF-MI transition is of the second order, the phase boundary curve is exact in infinite dimensions. Hence, if we expand the equation for this phase boundary curve, obtained by the PMFA up to the third order of  $t$ , it must agree with our proposed strong-coupling expansion in infinite dimensions. The results of the expansion of this phase boundary curve obtained by PMFA (Eqs. 30 and 46

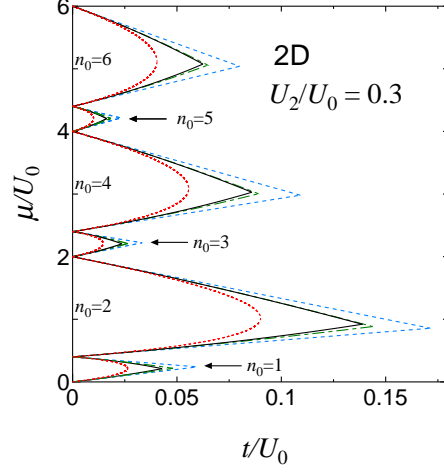


FIG. 2. (Color online) Same plot as that in Fig. 1 for  $U_2/U_0 = 0.3$  in 2D.

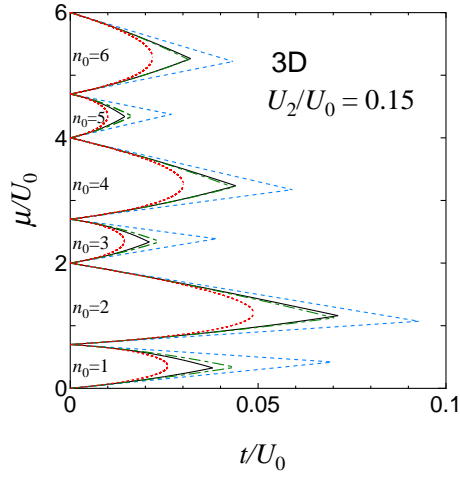


FIG. 3. (Color online) Same plot as that in Fig. 1 for  $U_2/U_0 = 0.15$  in 3D.

of Ref. [16]) are as follows:

$$\begin{aligned} \mu_{\text{even}}^{\text{part}} = & U_0 n_0 - \frac{n_0 + 3}{3} zt - \frac{n_0(n_0 + 3)}{9} \frac{(zt)^2}{U_0 + 2U_2} \\ & - \frac{n_0(n_0 + 3)(2n_0 + 3)}{27} \frac{(zt)^3}{(U_0 + 2U_2)^2} \end{aligned} \quad (17)$$

$$\begin{aligned} \mu_{\text{even}}^{\text{hole}} = & U_0(n_0 - 1) - 2U_2 + \frac{n_0}{3} zt + \frac{n_0(n_0 + 3)}{9} \frac{(zt)^2}{U_0 + 2U_2} \\ & + \frac{n_0(n_0 + 3)(2n_0 + 3)}{27} \frac{(zt)^3}{(U_0 + 2U_2)^2} \end{aligned} \quad (18)$$

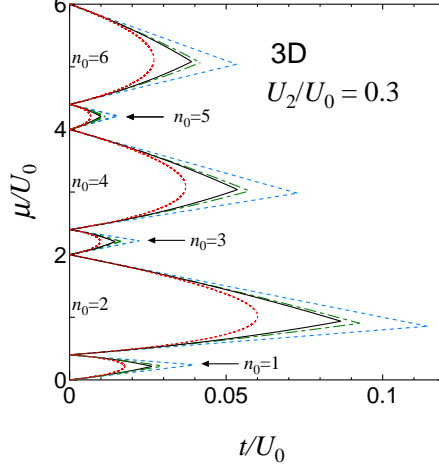


FIG. 4. (Color online) Same plot as that in Fig. 1 for  $U_2/U_0 = 0.3$  in 3D.

for even MI lobes, and

$$\begin{aligned} \mu_{\text{odd}}^{\text{part}} = & U_0 n_0 - 2U_2 - \frac{n_0 + 3}{3} zt - \frac{n_0 + 1}{9} \left[ \frac{n_0 + 2}{U_0 - 2U_2} + \frac{4}{5} \left( \frac{n_0 - 1}{U_0 + U_2} + \frac{n_0 + 4}{3U_2} \right) \right] (zt)^2 \\ & - \frac{n_0 + 1}{27} \left\{ \frac{(n_0 + 2)(2n_0 + 3)}{(U_0 - 2U_2)^2} + \frac{4}{25} \frac{(n_0 - 1)(9n_0 + 1)}{(U_0 + U_2)^2} - \frac{4}{25} \frac{(n_0 + 4)(n_0 - 11)}{(3U_2)^2} \right. \\ & \left. + \frac{24}{25} \frac{(n_0 + 4)(3n_0 + 2)}{3U_2(U_0 + U_2)} + \frac{8}{5} \frac{(n_0 + 2)(2n_0 + 3)}{(U_0 + U_2)(U_0 - 2U_2)} \right\} (zt)^3 \end{aligned} \quad (19)$$

$$\begin{aligned} \mu_{\text{odd}}^{\text{hole}} = & U_0(n_0 - 1) + \frac{n_0 + 2}{3} zt + \frac{n_0 + 2}{9} \left[ \frac{n_0 + 1}{U_0 - 2U_2} + \frac{4}{5} \left( \frac{n_0 - 1}{3U_2} + \frac{n_0 + 4}{U_0 + U_2} \right) \right] (zt)^2 \\ & + \frac{n_0 + 2}{27} \left\{ \frac{(n_0 + 1)(2n_0 + 3)}{(U_0 - 2U_2)^2} + \frac{4}{25} \frac{(n_0 + 4)(9n_0 + 26)}{(U_0 + U_2)^2} - \frac{4}{25} \frac{(n_0 - 1)(n_0 + 14)}{(3U_2)^2} \right. \\ & \left. + \frac{24}{25} \frac{(n_0 - 1)(3n_0 + 7)}{3U_2(U_0 + U_2)} + \frac{8}{5} \frac{(n_0 + 1)(2n_0 + 3)}{(U_0 + U_2)(U_0 - 2U_2)} \right\} (zt)^3 \end{aligned} \quad (20)$$

for odd MI lobes. These are in exact agreement with our proposed strong-coupling expansion. Specifically, we find that the solutions of the equations  $E_{\text{MI,even(odd)}}(n_0) = E_{\text{part(hole)}}^{\text{part(hole)}}(n_0)_{\text{def,even(odd)}}$  are the same as Eqs. 17–20 under the limit of  $z \rightarrow \infty$  and  $t \rightarrow 0$  keeping  $zt$  as a constant.

### E. Critical value of $t$ at the peak of the Mott lobe

In this subsection, we examine the critical value of the hopping parameter  $t$  at the peak of the Mott lobe ( $t_C$ ), at which the upper branch of  $t$ - $\mu$  curve for the phase boundary converges with its lower branch. In addition, the dependence of  $t_C$  on  $U_2$  indicates the range of the application of our proposed expansion for the third order of  $t$ .

Figure 5 shows the dependence of  $t_C$  on  $U_2$  for an Mott lobe with an even number of bosons ( $n_0 = 2$ ). The curves of infinite dimensions obtained by PMFA and those obtained by the strong-coupling expansion up to the third order of  $t$  (see the previous subsection) are smoothly increasing functions of  $U_2$ . The results obtained by strong-coupling expansion up to the third order of  $t$  in 2D and 3D show a similar  $U_2$  dependence of  $t_C$  when  $U_2/U_0$  is large. However, they show a uncommon behavior for  $U_2/U_0 \sim 0.1$ , and the curves stop at small  $U_2/U_0$  because we cannot find  $t_C$  as the upper and lower branches of the  $t$ - $\mu$  curve no longer converge. Such a situation also occurs for a greater  $n$ . Because  $U_2$  stabilizes the MI phase with even boson filling against the SF phase, the results of PMFA and the strong-coupling expansion of infinite dimensions are in accordance with physical intuition. At the small  $U_2$  regime is hazardous for strong-coupling expansion of finite dimensions because a few denominators of the expansion have  $3U_2$  which corresponds to the spin-excitation energy  $[E(S = 2) - E(S = 0)]$  of an intermediate state that appears in perturbative calculation. On the other hand, the terms including  $3U_2$  disappear in the denominators of strong-coupling expansion in infinite dimensions.

Figure 6 shows the same plot that in as Fig. 5 for a Mott lobe with an odd number of bosons ( $n_0 = 1$ ). The parameter  $t_C$  obtained by PMFA is a smoothly decreasing function of  $U_2$  and becomes zero at  $U_2/U_0 = 0.5$ , at which the Mott lobes with odd boson fillings disappear. Moreover, the other curves also show that  $t_C$  is a decreasing function of  $U_2$  when  $U_2/U_0$  is large. However,  $t_C$  is a rapidly increasing function of  $U_2$  for  $U_2/U_0 < 0.1$  and  $t_C = 0$  at  $U_2/U_0 = 0$ . This is also because a few denominators have  $3U_2$  in strong-coupling expansion up to the third order of  $t$  (here, not only in 2D or 3D but also in infinite dimensions). Therefore,  $t_C \rightarrow 0$  when  $U_2/U_0 \rightarrow 0$ , so that the two branches of  $t$ - $\mu$  curve converge. For a greater  $n_0$ ,  $t_C$  cannot be found even for a small  $U_2/U_0$ , same as that for the above-mentioned case of even  $n_0$  (figure not shown). These problems can be solved only by expanding  $\mu$  in the infinite orders of  $t$ , same as that in PMFA. A similar problem can also occur for  $U_2/U_0 \approx 0.5$  in 1D because a few terms in the expansion include  $U_0 - 2U_2$  in their denominator (see subsection G). The parameter  $t_C$  may be an increasing (decreasing) function of  $U_2$  when the Mott lobe has an even (odd) boson filling, and strong-coupling expansion can be depended upon only in a large (an intermediate)  $U_2$  region.

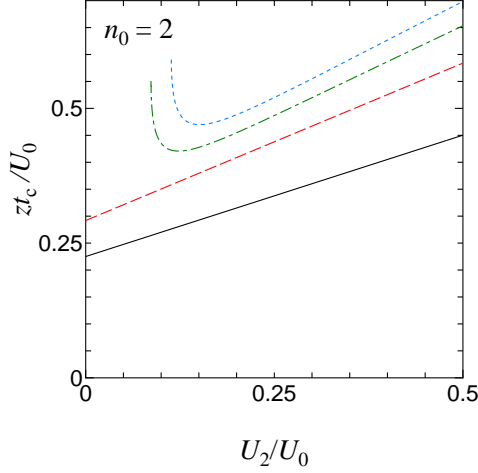


FIG. 5. (Color online) Dependence of  $t_C$  on  $U_2$  for the Mott lobe with an even boson filling  $n_0 = 2$ . The blue short-dashed, green dot-dashed, and red long-dashed curves show the results obtained by strong coupling expansion up to the third order of  $t$  in 2D, 3D, and infinite dimensions, respectively. The solid curves show the results obtained by PMFA.

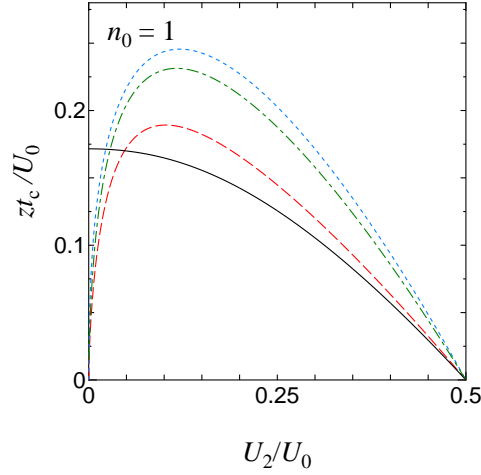


FIG. 6. (Color online) Same plot as that shown in Fig. 5 for the Mott lobe with an odd boson filling  $n_0 = 1$ .

### F. Extrapolation methods

Our expansion up to the third order of  $t$  has a few problems. For example, the phase boundary curve including the value of  $t_C$  does not completely converge. From a qualitative point of view, expansion does not provide an appropriate scaling form of the phase boundary curve near  $t_C$ . However, in this subsection, we attempt two extrapolation methods toward

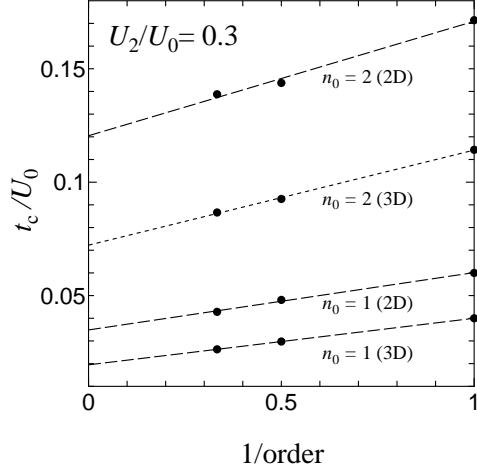


FIG. 7. Solid circles show  $t_C$  obtained by our proposed strong-coupling expansion at the first, second, and third orders of  $t$  for  $U_2/U_0 = 0.3$  in 2D and 3D. The data are indicated for  $n_0 = 1$  Mott lobes and  $n_0 = 2$  Mott lobes. The horizontal axis shows the inverse of the order of expansion ( $1/\text{order} = 1/3, 1/2, 1$  for third, second, and first orders, respectively). The dashed lines are least-square linear fits for the solid circles.

an infinite-order theory to improve our phase diagram.

### 1. Linear fit of $t_C$

Critical-point extrapolation, which was proposed in Ref. [31], is a simple method involving the use of least-squares fit to the best straight line.

Figure 7 shows the critical point  $t_c$  at the peak of the Mott lobe for each order ( $m$ ) of strong-coupling expansion. The data of  $t_C$  lie approximately on the straight line. The data can be extrapolated to the infinite order ( $1/m \rightarrow 0$ ) by using the least-squares fit for the best straight line. Specifically, the point crossed by the straight line over a vertical axis is the resulting  $t_C$  by extrapolation to the infinite order. Table I shows the fitting data to the infinite order. The error [41] of the fitting is very small when  $U_2/U_0$  is large, at which strong coupling can be more reliable than that in a small  $U_2/U_0$  regime (which is consistent with the dependence of  $t_C$  on  $U_2$  explained in the previous subsection).

## 2. Extrapolation of the phase boundary curves

Our proposed phase-boundary curve obtained by expansion up to the third order of  $t$  has a cusp at the peak of the Mott lobe. However, it can be assumed that chemical potential has the following power-law scaling behavior near  $t_C$ , similar to that the spinless BH model in 2D or 3D.

$$\mu = A(t) \pm B(t)(t_C - t)^{z\nu}. \quad (21)$$

The following fitting method is called as chemical-potential fitting [31, 38]. Here,  $A(t) \approx a + bt + ct^2 + dt^3$  and  $B(t) \approx \alpha + \beta t + \gamma t^2 + \delta t^3$  are the regular functions of  $t$ . The parameter  $z\nu$  is the critical exponent of our model. By using expansion up to the third order of  $t$ , we immediately determine  $a, b, c$ , and  $d$  by setting  $A(t) = [\mu^{\text{part}}(t) + \mu^{\text{hole}}(t)]/2$ . On the other hand, we assume that the value of  $z\nu$  to determine  $B(t)$ . If we assume the scaling behavior is the same as that of the spinless BH model or equivalent to that of the  $(d+1)$ -dimensional  $XY$  model,  $z\nu \simeq 2/3$  for  $d = 2$  and  $z = 1/2$  for  $d > 2$  in the  $d$ -dimensional spin-1 BH model [42]. By setting  $\delta = 0$ , we obtain  $\alpha, \beta, \gamma$ , and  $t_C$  by comparing the Taylor expansion of  $t$  in  $B(t)(t_C - t)^{z\nu}$  with  $[\mu^{\text{part}}(t) - \mu^{\text{hole}}(t)]/2$ . The values of  $t_C$  are shown in Table I and the phase-boundary curves obtained by the above fitting are shown in Fig. 8 for 2D and Fig. 9 for 3D.

We can combine these two fitting methods to obtain the phase-boundary curve. Specifically, we can use the value of  $t_C$  obtained by the least-square fit to compare  $B(t)(t_C - t)^{z\nu}$  with  $[\mu^{\text{part}}(t) - \mu^{\text{hole}}(t)]/2$ . Here we can include the  $\delta t^3$  term in  $B(t)$ . The obtained phase-boundary curves are also plotted in Figs. 8 and 9. Practically, the phase-boundary curves plotted by these two fitting methods are similar, especially in 2D.

The phase-boundary curves plotted by these two fitting methods are more similar to those obtained by PMFA in 3D than those in 2D, as expected.

## G. One dimension

In 1D, the MI phase has very rich spin structures; however, our strong-coupling expansion is based on the spin-singlet (spin-nematic) states for even (odd) boson filling. In particular, for odd boson fillings, the ground state may be the spin dimerized state in the wide parameter regime. Hence, our results cannot be directly applied to 1D especially for odd boson fillings.

The range of the  $U_2/U_0$  value, in which we can obtain  $t_C$  at the peak of the Mott lobe, is more limited compared to those in 2D or 3D. For instance,  $U_2/U_0 \geq 0.255$  is needed to obtain  $t_C$  for  $n_0 = 2$  Mott lobe.

Nevertheless, we briefly examine the phase diagram because most of the numerical simulations have been performed for 1D so far. Figures 10 and 11 show the phase diagrams obtained by our proposed strong-coupling expansion up to the third order of  $t$  for  $U_2/U_0 = 0.3$  and  $U_2/U_0 = 0.4$ , respectively. In Fig. 11, the upper and lower branches of the  $n_0 = 1$  Mott lobe do not converge, and a closed phase-boundary curve cannot be obtained. In addition, Figs 10 and 11 also show the convergence of the strong-coupling expansion from the first to the third order of  $t$ , and we find that the convergence is not excellent compared with that in 2D or 3D (Figs. 1–4).

For  $U_2/U_0 = 0.3$ , the agreement with the phase diagram plotted by DMRG in Ref. [22] is not excellent but is satisfactory to some extent for the Mott lobe with even boson fillings. For example,  $t_C/U_0 = 0.327$  by our proposed strong-coupling expansion, and  $t_C/U_0 \simeq 0.47$  by DMRG (the author's reading off Fig. 1 in Ref. [22]). However, for  $U_2/U_0 = 0.4$ , the agreement with QMC data, shown in Fig. 1 of Ref. [25] is not satisfactory:  $t_C/U_0 = 0.422$  in our proposed strong-coupling expansion and  $t_C/U_0 \simeq 0.7$  in the QMC data. In general,

TABLE I. List of the critical points  $t_c/U_0$ .

| $n_0$ | $U_2/U_0$ | Two dimensions      |                     |             | Three dimensions    |                     |             |
|-------|-----------|---------------------|---------------------|-------------|---------------------|---------------------|-------------|
|       |           | $(t_c/U_0)_{3rd}^a$ | $t_c/U_0^b$         | $t_c/U_0^c$ | $(t_c/U_0)_{3rd}^a$ | $t_c/U_0^b$         | $t_c/U_0^c$ |
| 1     | 0.2       | 0.0568              | $0.0400 \pm 0.0010$ | 0.0482      | 0.0353              | $0.0224 \pm 0.0020$ | 0.0269      |
|       | 0.3       | 0.0429              | $0.0349 \pm 0.0021$ | 0.0365      | 0.0263              | $0.0195 \pm 0.0001$ | 0.0202      |
|       | 0.4       | 0.0234              | $0.0205 \pm 0.0015$ | 0.0201      | 0.0143              | $0.0115 \pm 0.0004$ | 0.0110      |
| 2     | 0.2       | 0.1221              | $0.1019 \pm 0.0218$ | 0.1084      | 0.0758              | $0.0615 \pm 0.0078$ | 0.0607      |
|       | 0.3       | 0.1388              | $0.1205 \pm 0.0068$ | 0.1208      | 0.0866              | $0.0722 \pm 0.0021$ | 0.0683      |
|       | 0.4       | 0.1567              | $0.1389 \pm 0.0010$ | 0.1356      | 0.0978              | $0.0827 \pm 0.0009$ | 0.0766      |
| 3     | 0.2       | 0.0313              | $0.0216 \pm 0.0015$ | 0.0266      | 0.0195              | $0.0121 \pm 0.0016$ | 0.0149      |
|       | 0.3       | 0.0236              | $0.0190 \pm 0.0009$ | 0.0201      | 0.0145              | $0.0106 \pm 0.0001$ | 0.0111      |
|       | 0.4       | 0.0129              | $0.0113 \pm 0.0008$ | 0.0111      | 0.0079              | $0.0063 \pm 0.0002$ | 0.0061      |
| 4     | 0.2       | 0.0754              | $0.0611 \pm 0.0150$ | 0.0669      | 0.0469              | $0.0368 \pm 0.0057$ | 0.0375      |
|       | 0.3       | 0.0859              | $0.0726 \pm 0.0057$ | 0.0747      | 0.0536              | $0.0435 \pm 0.0021$ | 0.0422      |
|       | 0.4       | 0.0971              | $0.0840 \pm 0.0007$ | 0.0839      | 0.0606              | $0.0498 \pm 0.0003$ | 0.0474      |

<sup>a</sup> Data obtained by third-order strong-coupling expansion

<sup>b</sup> Data obtained by least-squares fit based on the strong-coupling expansion.

<sup>c</sup> Data obtained by chemical-potential fit.



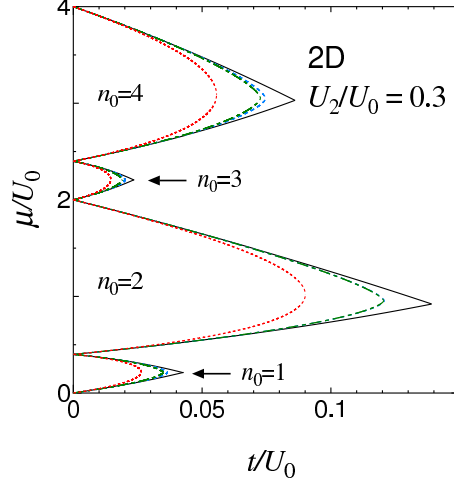


FIG. 8. (Color on line) Phase diagram obtained by strong-coupling expansion up to the third order of  $t$  (solid curve) for  $U_2/U_0 = 0.3$  in 2D and its extrapolation to infinite order of  $t$ . The blue dashed (green dot-dashed) curve shows the chemical-potential fitting without (with) the least-squares fit for  $t_C$ . The blue dashed and green dot-dashed curves are similar to each other. Especially, the two curves for the  $n_0 = 2$  Mott lobe are indistinguishable in the figure. The red dotted curve shows the results of PMFA. The area of the Mott lobe is smallest in PMFA.

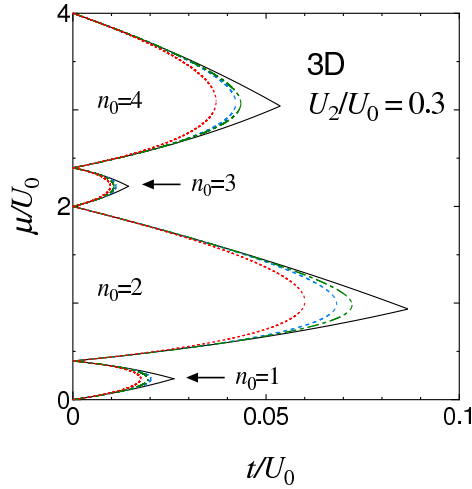


FIG. 9. Same plot as that in Fig. 8 for 3D.

a larger  $t$  is required to realize the SF phase for large  $U_0$  and/or  $U_2$ , where the higher-order terms of  $t$  become more prominent. Therefore, we may have to expand up to fourth- or even higher-order terms to reduce the discrepancy. Otherwise, we may have to assume another MI phase such as a nematic MI phase, if the MI phase is more stable than the singlet MI

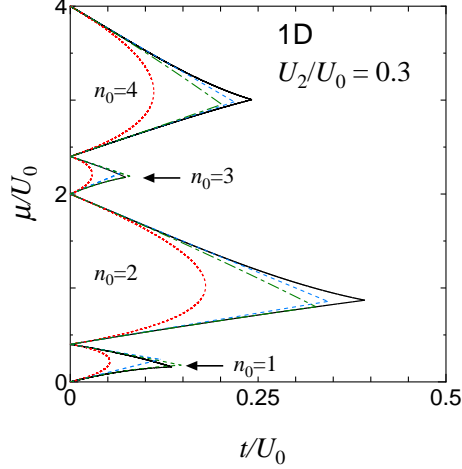


FIG. 10. (Color online) 1D phase diagram at  $U_2/U_0 = 0.3$ . The solid curves show the results obtained by strong-coupling expansion up to the third order of  $t$ . Results up to the first order (second order) of  $t$  are also shown by the blue dashed (green dot-dashed) curve. The red dotted curve shows the results obtained by PMFA.

phase that is assumed in this study.

Figures 10 and 11 also show the results obtained by PMFA, which may be inaccurate in 1D. We find that the difference between strong-coupling expansions is very large.

As mentioned in the previous subsection, we also attempt to extrapolate our results to infinite order of  $t$ . In the least-squares fit of  $t_C$ , we cannot improve our results as extensively; for instance,  $t_C/U_0 = 0.433 \pm 0.030$  at  $U_2/U_0 = 0.4$  for the  $n_0 = 2$  Mott lobe. We also perform the chemical-potential fit by assuming the Kosterlitz-Thouless form, as mentioned in Ref. [31]. However, we cannot obtain successful fitting results.

## H. Summary and Discussion

In this study, we have performed strong-coupling expansion of the hopping parameter  $t$  to analytically obtain the phase diagram. Under the limit of infinite dimensions, the  $t$ - $\mu$  phase boundary curves are consistent with the exact results obtained by PMFA. On the whole, the convergence of the phase-boundary curves from the first to the third order is excellent in 2D or 3D. The dependence of  $t_C$  on  $U_2$  at the peak of the Mott lobe indicates the validity of our expansion. We have attempted to extrapolate our results up to the infinite order of  $t$  by using the least-squares fitting and chemical-potential fittings of  $t_C$ . The results are more

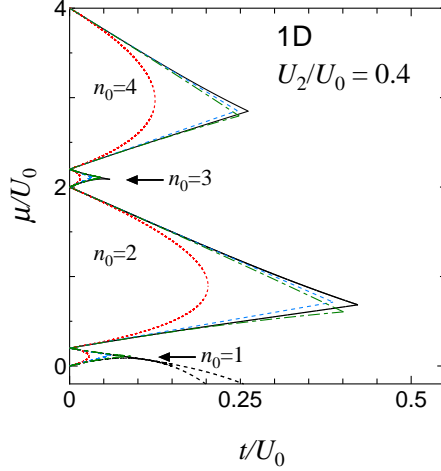


FIG. 11. (Color online) Similar plot as that of Fig. 10 for  $U_2/U_0 = 0.4$  in 1D, except for the dashed curves for the  $n_0 = 1$  Mott lobe. For  $n_0 = 1$ , the upper and lower branches of the Mott lobe do not converge in the third order of  $t$ , and we cannot obtain a closed phase-boundary curve.

equal to those of PMFA in 3D than those in 2D, as expected. In 1D, we have compared the phase boundary curves obtained by strong-coupling expansion with those obtained by numerical simulations. For  $U_2/U_0 = 0.3$ , the agreement with the result obtained by DMRG is not excellent but is satisfactory to some extent.

Our proposed strong-coupling expansion depends on the  $t = 0$  ground state. However, the MI phase can be more complicated. We have to consider the dimerized-spin phase for odd boson fillings, which can be the ground state in 1D. In addition, we should consider the nematic spin phase for even boson fillings, which can be the ground state for weak  $U_2$ . To analytically determine the complete phase diagram, we have to include such spin phases in our strong-coupling expansion. This still remains a problem in future.

## Appendix: Energies of insulating states

In Sec. III A, we have introduced three degenerate ground states at  $t = 0$  when the boson filling  $n_0$  is odd.

$$\Psi_{\text{odd}} = \prod_k |n_0, 1, 0\rangle_k, \quad (\text{A.1})$$

$$\Psi_{\text{ferro}} = \prod_k |n_0, 1, \pm 1\rangle_k \quad (\text{A.2})$$

$$\begin{aligned} \Psi_{\text{dimer}} = \prod_{\langle i,j \rangle} \frac{1}{\sqrt{3}} & \left( |n_0, 1, 1\rangle_i \otimes |n_0, 1, -1\rangle_j \right. \\ & \left. - |n_0, 1, 0\rangle_i \otimes |n_0, 1, 0\rangle_j + |n_0, 1, -1\rangle_i \otimes |n_0, 1, 1\rangle_j \right). \end{aligned} \quad (\text{A.3})$$

By using the Rayleigh-Schrödinger perturbation theory [32], we obtain the energies as follows:

$$\frac{E_{\text{ferro}}}{N} = -zt^2 \left[ \frac{7}{75} \frac{(n_0 - 1)(n_0 + 4)}{U_0 + 4U_2} + \frac{2}{15} \frac{2n_0^2 + 6n + 7}{U_0 + U_2} \right] + O(t^4), \quad (\text{A.4})$$

$$\begin{aligned} E_{\text{dimer}} = -\frac{t^2}{27} & \left\{ \frac{16}{5} \frac{(n_0 - 1)(n_0 + 4)}{U_0 + 4U_2} - 2 \frac{2n_0^2 + 6n + 7}{U_0 + U_2} + 8 \frac{(n_0 + 1)(n_0 + 2)}{U_0 - 2U_2} \right. \\ & \left. + z \left[ 4 \frac{(n_0 - 1)(n_0 + 4)}{U_0 + 4U_2} + 2 \frac{2n_0^2 + 6n + 7}{U_0 + U_2} + \frac{(n_0 + 1)(n_0 + 2)}{U_0 - 2U_2} \right] \right\} + O(t^4). \end{aligned} \quad (\text{A.5})$$

$E_{\text{MI,odd}}$  is obtained as specified in Eq. 12, up to the third order of  $t$ . Comparing Eq. A.4 with Eq. 12, we can easily find that  $E_{\text{ferro}} > E_{\text{MI,odd}}$  for  $U_2 > 0$  at any value of  $n_0$  and  $z = 2d$ . By comparing Eq. A.5 with Eq. 12, we also find that  $E_{\text{dimer}} > E_{\text{MI,odd}}$  for  $U_2 > 0$  in 2D or larger dimensions. On the other hand, in 1D,  $E_{\text{dimer}} > (<)E_{\text{MI,odd}}$  for  $U_2 < (>)U_0/6 > 0$  when  $n_0 = 1$  and  $E_{\text{dimer}} < E_{\text{MI,odd}}$  for  $U_2 > 0$  when  $n_0 \geq 3$ .

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